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INVESTIGATION OF STABILITY IN CERTAIN PROBLEMS OF NON-LINEAR MECHANICS*

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The generalized Liapunov second method /1/ is used to investigate the stability of resonance modes in a non-linear multifrequency system. The results are used to study the oscillations of a pendulum whose point of suspension performs small harmonic two-frequency motions in resonance with the natural oscillation of the pendulum and, also, the two-frequency problem of the motion of a satellite relative to the centre of mass. In estimating the small denominators in these problems no reduction to systems with a lower number of frequencies was achieved.

Consider the equation

$$x^{\cdot\cdot}(t)+\omega^2\sin x=\mu f(x,\ x^{\cdot},\ \omega_1 t,\ \ldots,\ \omega_n t),\ 0<\mu\ll 1 \tag{1}$$

which describes many important practical oscillation processes such as the motion of a satellite in a gravitational field of force, the vibration of the mechanical structure of an accelerator with rigid focusing, etc.

Linear and non-linear equations of the form (1) with small non-linear perturbations have been investigated by many authors. A detailed investigation of the action of external periodic forces on oscillaing systems close to linear was carried out over a long time interval $T=O(\mu^{-1})$, and questions of the stability of resonance amplitudes in the case of parametric resonance were considered in the single-frequency case /2, 3/.

Below, Eq.(1) is considered on the assumption that the natural oscillation frequencies and the external forces are of comparable magnitude. In this case it reduces to a multi-frequency system of first-order differential equations of the standard Bogoliubov form. The small denominators that occur, inherent in multifrequency systems, are estimated in terms of the magnitude of the neighbourhood of the point investigated for stability.

When $\mu=0$, Eq.(1) is integrated in elliptic functions, whose general solution defines the oscillation

$$\alpha=2\ \arcsin\ (k\ \mathrm{sn}\ (2\pi^{-1}K(k)\ \varphi,\ k))$$

or rotation

$$a=2\ \mathrm{am}\ (\pi^{-1}K(k)\ \varphi,\ k)$$

where

$$\varphi=\omega_0(k)\ (t+t_0),\ t_0=\mathrm{const}$$

$$\omega_0(k)=\frac{\pi\omega}{2K(k)}\ \text{ in the case of oscillations}$$

$$\omega_0(k)=\frac{\pi\omega}{kK(k)}\ \text{ in the case of rotation}$$

$K(k)$ is the complete elliptic integral of the first kind and $\omega_0(k)$ is the frequency of natural motions.

Consider the case of oscillations (rotation can be investigated similarly). Since $\alpha_{\max}=2\ \arcsin k$, then it is convenient to select k, as the variable that defines the oscillation amplitude, and to select φ as the phase.

We will transform Eq.(1) into a multi-oscillation system in terms of the variables k, φ in the standard Bogoliubov form. We assume that $f(x,\ y,\ \bar{z})=f(x,\ y,\ z_1,\ \ldots,\ z_{n+1})$ is a 2π periodic function of the variables $z_1,\ \ldots,\ z_{n+1}$, and $[n/2]+2$ times continuously differentiable and

$$f_0(x,y)=\frac{1}{(2\pi)^n}\int_0^{2\pi} f(x,y,\bar{z})\,d\bar{z}=0$$

We substitute into Eq.(1) the variables

$$x=\alpha\ (k,\ \varphi),\ x^{\cdot}=\omega_0\ (k)\ \alpha_\varphi{}'\ (k,\ \varphi) \tag{2}$$

and introduce the action integral

$$I(k)=\frac{\omega_0(k)}{2\pi}\int_0^{2\pi}\alpha_\varphi{}^2(k,\varphi)\,d\varphi,\quad I'(k)=D(k)$$

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For the new variables k, φ we obtain a multifrequency system of ordinary differential equations in standard Bogoliubov form

$$
\begin{aligned}
&k^{\cdot} = \mu f(\alpha(k, \varphi), \omega_0(k)\alpha_\varphi'(k, \varphi), \varphi_1, \ldots, \varphi_n) \times \\
&\quad \alpha_\varphi'(k, \varphi)/D(k) = \mu F(k, \overline{\varphi}) \\
&\varphi^{\cdot} = \omega_0(k) - \mu f(\alpha(k, \varphi), \omega_0(k)\alpha_\varphi'(k, \varphi), \varphi_1, \ldots \\
&\quad \varphi_n)\, \alpha_k'(k, \varphi)/D(k) = \omega_0(k) + \mu\Phi(k, \overline{\varphi}) \\
&\varphi_i^{\cdot} = \omega_i, \ \varphi_i = \omega_i t, \ i = 1, \ldots, n, \ \overline{\varphi} = (\varphi, \varphi_1, \ldots, \varphi_n)
\end{aligned} \tag{3}
$$

The vector of the frequencies $\Omega = (\omega_0(k), \omega_1, \ldots, \omega_n)$ consists of the angular frequency $\omega_0(k)$ and the perturbing-force frequencies.

We call k_0 the point of system resonance, if there exists an integral vector $\bar{p} = (p_0, p_1, \ldots, p_n)$ such that

$$
p_0\omega_0(k_0) + p_1\omega_1 + \ldots + p_n\omega_n = 0
$$

To investigate the Liapunov stability of some resonance amplitude, we apply the generalized Liapunov second method /1/.

As the unperturbed Liapunov function we take $V_0 = |k - k_0|$ and consider the segments $\eta < |k - k_0| < \varepsilon$ where $0 < \varepsilon < \varepsilon_0$ and ε_0 (by virtue of the properties of the function $\omega_0(k)$) does not exceed the distance from k_0 to the nearest resonance amplitude. Then the ε-neighbourhood of the point for $0 < \varepsilon < \varepsilon_0$ does not contain other resonance values.

The perturbed Liapanov function for $\eta < |k - k_0| < \varepsilon$ is

$$
\begin{aligned}
&V(k, \overline{\varphi}) = V_0(k) + \mu V_1(k, \overline{\varphi}) \\
&V_1(k, \overline{\varphi}) = \sum_{m \neq 0} i\, \frac{F_m(k)\operatorname{sign}(k - k_0)}{(\Omega, m)} e^{i(m, \overline{\varphi})}
\end{aligned} \tag{4}
$$

where $F_m(k)$ are the coefficients of the Fourier series of the function $F(k, \overline{\varphi})$.

The function $V_1(k, \overline{\varphi})$ in the ring indicated exists and is bounded, since by virtue of the assumptions made the function $F(k, \overline{\varphi})$ can be expanded in absolutely and uniformly convergent Fourier series and $F_0(k) = 0$, andfor the small denominators (Ω, m) the estimate

$$
\begin{aligned}
&m_0\omega_0(k) + \sum_1^n m_i\omega_i = m_0(\omega_0(k) - \omega_0(k_0)) \approx \\
&\omega_0'(k_0)|k - k_0| = O(\varepsilon, \eta)\,\omega_0'(k_0), \quad \omega_0'(k_0) < 0
\end{aligned}
$$

holds by virtue of the properties of the function $\omega_0(k)$.

If we differentiate the function (4) along the solution of system (3) we obtain

$$
\left.\frac{dV(k, \bar{\varphi})}{dt}\right|_{(3)} = O(\mu^2) \tag{5}
$$

All the conditions of Theorem 2 of /1/ are thus satisfied for system /3/ and the Liapunov function (4), and for the time during which the solution $k(t)$ remains in the ε-neighbourhood of point k_0 the estimate $T = O(\mu^{-2})$ holds.

All these estimates hold for the first resonances, as long as the difference between two adjacent resonances does not become a quantity of order μ.

Example 1. Let us use the results obtained to investigate the stability of the resonance amplitude of a pendulum whose point of suspension performs plane oscillations as given by

$$
x = a\cos\nu_1 t, \quad y = b\sin(\nu_2 t + \chi)
$$

where a, b, χ are certain constants.

Let ψ be the angle of deflection from the lower position of equilibrium, $\omega = \sqrt{g/l}$ be the natural oscillation frequency of the pendulum, and l the length of the thread. We assume that $a/l \ll 1$. This means that the point of suspension performs small oscillations. The pendulum equation of motion has the form

$$
\begin{aligned}
&\psi^{\cdot\cdot} + \sin\psi = \mu(\cos\varphi_1\cos\psi + \delta\sin\varphi_2\sin\psi) \\
&\omega_1 = \frac{\nu_1}{\omega}, \ \omega_2 = \frac{\nu_2}{\omega}, \ \varphi_1 = \nu_1 t \\
&\varphi_2 = \nu_2 t + \chi, \ \delta = \frac{b}{a}\left(\frac{\omega_2}{\omega_1}\right)^2, \quad \tau = \omega t, \ \mu = \frac{a}{l}|\omega_1^2
\end{aligned}
$$

Using the substitution (2), we obtain a three-frequency system in standard Bogoliubov form

$$
\begin{aligned}
&k^{\cdot} = \mu\left[\frac{1}{2}\operatorname{cn}\frac{2K(k)}{\pi}\varphi\left(1 - 2k^2\operatorname{sn}^2\frac{2K(k)}{\pi}\varphi\right)\cos\varphi_1 + \right. \\
&\left. k\operatorname{sn}\frac{2K(k)}{\pi}\varphi\operatorname{cn}\frac{2K(k)}{\pi}\varphi\operatorname{dn}\frac{2K(k)}{\pi}\varphi\sin\varphi_2\right] \\
&\varphi^{\cdot} = \frac{\pi}{2K(k)} + \mu\Phi(k, \overline{\varphi}), \ \varphi_1^{\cdot} = \omega_1, \ \varphi_2^{\cdot} = \omega_2
\end{aligned} \tag{6}
$$

We expand the right side of the first of Eqs.(6) in a Fourier series

$$F(k,\overline{\varphi}) = \sum_n b_n(k)\,[\cos(n\varphi + \varphi_1) + \cos(n\varphi - \varphi_1)] +$$
$$\sum_m c_m(k)\,[\cos(m\varphi - \varphi_2) - \cos(m\varphi + \varphi_2)] + \sum_m a_m(k)\,[\sin(m\varphi + \varphi_2) +$$
$$\sin(m\varphi - \varphi_2)], \quad \int_0^{2\pi} F(k,\overline{\varphi})\,d\overline{\varphi} = 0$$

Noting that in this case the function $F(k,\overline{\varphi})$ is analytic in $\overline{\varphi}$. The coefficients $a_m(k), b_n(k), c_m(k)$ approach zero exponentially as m and $n \to \infty$. Consequently, when

$$\omega_0(k_0) = \frac{\omega_1}{n}, \quad \omega_0(k_0) = \frac{\omega_2}{m} \tag{7}$$

on the right side of the first of Eqs.(6) non-oscillating terms of the form $\mathrm{const}\cos\tau_0$, $\mathrm{const}\sin(\tau_0 - \chi)$, $\mathrm{const}\cos(\tau_0 - \chi)$ appear, i.e. resonance effects are present. In agreement with the above results, the point k_0 that satisfies one of Eqs.(7) is stable during the time $T = O(\mu^{-2})$.

Example 2. Let us investigate the stability of resonance modes in the problem of the oscillations and rotations of a satellite moving in an elliptic orbit in a central gravitational field. The system that describes the satellite motions is a two-frequency one, whose vector of frequencies consists of the satellite rotation frequency relative to the centre of mass and the frequency of its rotation the Earth.

Resonance oscillations and rotations of the satellite relative to its centre of mass in the orbital plane were earlier studied /4, 5/* (*See also Markeyev, A.P. Investigation of the stability of motion in some problems of celestial mechanis. Preprint Inst. Prikl. Matem. AN SSSR Moscow, 1970.) by the Krylov-Bogoliubov averaging method /2, 3/ with rigid constraints on resonance frequencies that reduce the two-frequency input systems to a single-frequency one.

The small oscillations of a satellite and resonance effects in the motion of the Moon were investigated in /6/.

Let the satellite principal axis of intertia be normal to the plane of the orbit. We denote the moments of inertia relative to that axis by B, and the moments of inertia relative to the two other principal axes by A, C $(A \geqslant C)$. Then, with an accuracy to within the ratio of the satellite dimensions to those of the orbit, the satellite equation of motion has the form /4/

$$\begin{aligned}&(1 + e\cos\theta)\,d^2\delta/d\theta^2 - 2e\sin\theta\,d\delta/d\theta + 3a^2\sin\delta = 4e\sin\theta\\ &a^2 = (A - C)/B \leqslant 1\end{aligned} \tag{8}$$

where δ is twice angle between the radius vector of the centre of mass and the axis of inertia, with the moment of inertia relative to it being C, e is the orbit eccentricity, and θ is the angular distance of the radius vector from the orbit perigee.

When $e = 0$, Eq.(8) reduces to the equation of the pendulum and defines the motion of a satellite in a circular orbit.

Consider the resonance case $e \ll 1$ when the orbit is close to circular. To within quantities $O(e^2)$, we have

$$\frac{d^2\delta}{d\theta^2} + 3a^2\sin\delta = ef\left(\theta, \delta, \frac{d\delta}{d\theta}\right)$$
$$f\left(\theta, \delta, \frac{d\delta}{d\theta}\right) = 4\sin\theta + 2\sin\theta\,\frac{d\delta}{d\theta} + 3a^2\cos\theta\sin\delta$$

We transform this equation to Bogoliubov standard form, using a resplacement of variables similar to (2)

$$\delta = \alpha(k, \varphi), \quad \varphi = \omega_0(k)(\theta + \theta_0), \quad \theta_0 = \mathrm{const}.$$

We obtain a system of standard form of type (3)

$$k^{\cdot} = eF(k, \varphi, \theta), \quad \varphi^{\cdot} = \omega_0(k) + e\Phi(k, \varphi, \theta), \quad \theta^{\cdot} = 1$$
$$F(k, \varphi, \theta) = f(\theta, \alpha(k, \varphi), \omega_0(k)\,\alpha_\varphi'(k, \varphi))\,\alpha_\varphi'(k, \varphi)/D(k)$$
$$\Phi(k, \varphi, \theta) = -f(\theta, \alpha(k, \varphi), \omega_0(k)\,\alpha_\varphi'(k, \varphi))\,\alpha_k'(k, \varphi)/D(k)$$

We expand the right sides of this system in Fourier series, and use simple geometry to obtain terms of the form

$$G(k, \theta), \; g_n(k)\sin(\theta + n\varphi), \; g_n'(k)\sin(\theta - n\varphi)$$

Note that

$$\begin{aligned}&g_n'(k)\sin(\theta - n\varphi) = -g_n'(k)\\ &\sin(\theta - n\omega_0(k)(\theta + \theta_0)) = -g_n'(k)\sin\theta_0\end{aligned} \tag{9}$$

when $\omega_0(k) = 1/n$

Thus when $\omega_0(k) = 1/n$ where n is a positive integer, we have on the right sides of the system slowly varying terms of the form $-g_n'(k)\sin\theta_0$, i.e. resonance effects are observed both in the case of oscillations and in the case of rotations. By the above reasoning resonances of the form (9) are stable with respect to the slow variable k during a time $T = O(\varepsilon^{-2})$.

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ON THE ASYMPTOTIC STABILITY AND INSTABILITY OF THE ZEROTH SOLUTION OF A NON-AUTONOMOUS SYSTEM*

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A non-autonomous set of differential equations with right side satisfying conditions for the existence of limit sets of differential equations /1, 2/ is considered. Theorems are proved on the limit behaviour of the solutions, on the asymptotic stability and instability of the zeroth solution of such a set in the presence of a Liapunov function with a derivative of constant sign. On the basis of these theorems, sufficient conditions are obtained for the asymptotic stability and instability of the zeroth equilibrium position of a non-autonomous mechanical system. A problem is solved on the asymptotic stabilization of a given three-axis orientation in space for a solid with variable moments of inertia.

1. Consider the following set of differential equations

$$x^{\cdot} = X(t, x) \quad (X(t, 0) \equiv 0) \tag{1.1}$$

where x and X are real n-vectors, the function $X(t, x)$ is defined in the domain $R^+ \times \Gamma$ ($R^+ = [0, +\infty[$, $\Gamma = \{\|x\| \leqslant H < +\infty\}$, $\|x\|$ is a certain norm in R^n) and satisfies conditions (A) from /1/: $X(t, x)$ is measurable in t for fixed x, and is continuous in x for fixed t; for any compact set $\Gamma_1 \subset \Gamma$ two local L_1-functions $h_1(t)$ and $h_2(t)$ exist such that for any $x, y \in \Gamma_1$ and $t \in R^+$

$$\|X(t, x)\| \leqslant h_1(t), \quad \|X(t, x) - X(t, y)\| \leqslant h_2(t)\|x - y\|$$

the function $h_1(t)$ is uniformly continuous in the mean on any segement $[\tau, \tau + 1] \subset R^+$, and the function $h_2(t)$ is bounded in the norm on $[\tau, \tau + 1]$, i.e.

$$\int_E h_1(t)\,dt \leqslant \varepsilon, \quad \int_\tau^{\tau+1} h_2(t)\,dt \leqslant \rho$$

for any measurable set $E \subset [\tau, \tau + 1]$ by a measure less than $\mu = \mu(\varepsilon, \Gamma_1) > 0$, and a certain number $\rho = \rho(\Gamma_1)$.

As is shown in /1/, conditions (A) guarantee the existence of solutions of (1.1), in the Caratheodory sense, and their uniqueness, the compactness (in weak L_1-topology) of the family of functions $\{X(t, x)\}$, satisfying these conditions, particularly the existence of limit functions $\varphi(t, x)$ to $X(t, x)$, the mutual continuity of the solutions of the initial system (1.1), and the solutions of the limit systems

$$x^{\cdot} = \varphi(t, x) \tag{1.2}$$

We note that a special case of conditions (A) is Lipschitz conditions in t and x, which

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